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Evers, J.J.M.

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Research memorandum

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LINEAR INFINITE HORIZON PROGRAMMING AND LEMKE'S COMPLEMENTARITY ALGORITHM FOR THE CALCULATION OF EQUILIBRIUM COMBINATIONS

by

J.J.M. Evers

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1 LINFAR INFINITE HORIZON PROGRAMMING

1.1 Introduction

This paper will present a summary of the most important results concerning linear ∞-horizon programming (3). However, we limit ourselves to a simplified version of these results. Further, a linearized Hansen and Koopmans model will be identified as a special case of the LP-model under consideration. Part two will demonstrate that Lemke's complementarity algorithm is applicable for the calculation of a so called equilibrium combination of the LP-problem.

We consider the LP-problem

$$\sup_{\substack{x(t)\\y(t)}\ t=1,2,\ldots} \sum_{t=1}^{\infty} \pi^t p'x(t) \quad \left|\quad \begin{aligned} &Bx(1)+y(1)=\rho f+Ax(0)\\ &Bx(t+1)-Ax(t)+y(t+1)=\rho^{t+1}f,\ t=1,2,\ldots\\ &x(t),y(t)\geqq 0,\ t=1,2,\ldots \end{aligned}\right. \tag{1.1.1}$$

and its dual form:

$$\inf_{\substack{u(t)\\v(t)}\ t=1,2,\ldots} x(0)'A'u(1)+\sum_{t=1}^{\infty} \rho^t f'u(t) \quad \left|\quad \begin{aligned} &B'u(t)-A'u(t+1)-v(t)=\pi^t \mathbf{p},\ t=1,2,\ldots\\ &u(t),v(t)\geqq 0,\ t=1,2,\ldots \end{aligned}\right. \tag{1.1.2}$$

where the quantities are specified as follows [1]:

- $x(t) \in R^n_+$, $t=1,2,\ldots$ a sequence of economic state variables

- $y(t) \in R^m_+$, $t=1,2,\ldots$ a sequence of primal slack variables

---

[1] $a\geqq b$ for vectors a,b denotes $a_i\geqq b_i$ for all i; $a\geq b$ denotes $a\geqq b$ and $a\neq b$; $a>b$ denotes $a_i>b_i$ for all i. The positive ortant of R^m is defined by $R^m_+ := \{a\in R^m \mid a\geqq 0\}$.

- $u(t) \in R_+^m$, $t=1,2,\ldots$ a sequence of dual variables
- $v(t) \in R_+^n$, $t=1,2,\ldots$ a sequence of dual slack variables
- $p \in R^n$ the objective vector
- $f \in R^m$ the vector of availabilities and requirements.
- A, B mxn-matrices
- ρ, π ρ the growth factor, π the discount factor such that $\rho\pi < 1$.

A non-negative sequence $(\hat{x}(t),\hat{y}(t)) \in R^{n+m}$, $t=1,2,\ldots$, which satisfies the equalities of the primal problem (1.1.1) is called primal optimal if there is no non-negative sequence $(x(t),y(t)) \in R^{n+m}$, $t=1,2,\ldots$ which satisfies the equalities of (1.1.1) as well as

$$\sum_{t=1}^{T} \pi^t p'x(t) \geqq \sum_{t=1}^{T} \pi^t p'\hat{x}(t) + \varepsilon, \quad T \geqq T^*,$$

for some $\varepsilon > 0$ and some period T^*.

Dual optimality may be introduced in a simular manner.

1.2 Directedness

It appears [3,ch.1] that for practical purposes the usual conditions which are imposed on ∞-horizon LP-problems, like non-negativity of at least one of the matrices A or B, are rather strong. A much weaker condition is found in the concept of directedness, which is defined in a primal and a dual form:

a) The LP-problem is called primal directed (P-directed) if every row vector $b_{i.}$ of matrix B, with a negative component, corresponds with a non-negative row vector $a_{i.}$ of matrix A, and with a non-negative component f_i of vector f.

b) The LP-problem is called <u>dual directed</u> (D-directed) if every column vector $a_{.j}$ of matrix A, with a negative components, corresponds with a non-negative column vector $b_{.j}$ of matrix B and with a non-positive component p_j of vector p.

Clearly, $B \geqq 0$ implies P-directedness and $A \geqq 0$ implies D-directedness.

Henceforth, we shall suppose that the LP-problem is P- or D-directed.

1.3 Some terms with respect to the existence of feasible solutions

A sequence of vectors $(x(t),y(t)) \in R_+^{n+m}$, $t=1,2,\ldots$ is called:

a) a <u>P-feasible solution</u> if it satisfies the equalities of the primal problem,

b) a <u>P-regular solution</u> if it is P-feasible and if in addition a positive vector $z \in R^m$ exists such that: $y(t) > \rho^t z$, $t=1,2,\ldots$

A sequence of vectors $(u(t),v(t)) \in R_+^{m+n}$, $t=1,2,\ldots$ is called:

c) a <u>D-feasible solution</u> if the equalities of the dual problem are satisfied,

d) a <u>D-regular solution</u> if the sequence is D-feasible and if in addition, a positive vector $w \in R^n$ exists such that: $v(t) \geqq \pi^t w$, $t=1,2,\ldots$

The LP-problem is called superregular if simultaneously P- and D-regular solutions exist and the systems

$$(B-\pi A)z < f, \; z > 0 \tag{1.3.1}$$

$$(B'-\frac{1}{\rho}A')w > p, \; w > 0 \tag{1.3.2}$$

both are solvable [1].

1.4 Necessary and sufficient conditions concerning feasibility and regularity when the LP-problem is P- or D-directed.

a) If the system

$$(B-\frac{1}{\rho}A)x \leqq f,\ x \geqq 0 \tag{1.4.1}$$

is solvable, then and only then an $x(0) \in R^n_+$ exists such that there is a P-feasible solution.

b) If the system

$$(B-\frac{1}{\rho}A)x < f,\ x \geqq 0 \tag{1.4.2}$$

is solvable, then and only then an $x(0) \in R^n_+$ exists such that there is a P-regular solution.

c) If the system

$$(B'-\pi A')u \geqq p,\ u \geqq 0 \tag{1.4.3}$$

is solvable, then and only then there is a D-feasible solution.

d) If the system

$$(B'-\pi A')u > p,\ u \geqq 0 \tag{1.4.4}$$

is solvable, then and only then there is a D-regular solution.

---

[1] This definition differs from the definition of [3,pag.31]. However in this simplified LP-problem where $\rho\pi < 1$ and where P- or D-directedness is always assumed, both definitions are equivalent.

e) If the systems (1.3.1) and (1.3.2) are solvable, then for every $\alpha \geqq \pi$ and $\beta \leqq \frac{1}{\rho}$ the systems

$$(B-\alpha A)x < f,\ x \geqq 0, \tag{1.4.5}$$

$$(B'-\beta A')u > p,\ u \geqq 0 \tag{1.4.6}$$

are solvable.

Since $\rho\pi < 1$, the consequence of 1.4-e is that the solvability of the systems (1.3.1) and (1.3.2) implies the solvability of the systems (1.4.1) to (1.4.4).

1.5 Some properties concerning optimality when the LP-problem is P- or D-directed.

a) If P- and D-regular solutions exist then:

- The supremum of the primal problem is equal to the infimum of the dual problem.
- The primal and the dual problem both possess an optimal solution.
- Feasible solutions $(x(t),y(t))$, $t=1,2,\ldots$ and $(u(t),v(t))$, $t=1,2,\ldots$ of the primal and dual problem resp. are both optimal, if and only if simultaneously:

$$v(t)'x(t)+u(t)'y(t) = 0,\ t=1,2,\ldots \tag{1.5.1}$$

$$u(t+1)'Ax(t) \to 0,\ t\to\infty. \tag{1.5.2}$$

b) If the LP-problem is superregular then, the properties mentioned in (a) are valid and, in addition, vectors $(\bar{x},\bar{y})$, $(\bar{v},\bar{u}) \in R_+^{n+m}$, exist such that every P-optimal $(\hat{x}(t),\ \hat{y}(t))$, $t=1,2,\ldots$ and every D-optimal $(\hat{u}(t),\ \hat{v}(t))$, $t=1,2,\ldots$ satisfies:

$$\left.\begin{aligned}(x(t),y(t)) &\leqq \rho^t(\bar{x},\bar{y})\\ (u(t),v(t)) &\leqq \pi^t(\bar{u},\bar{v})\end{aligned}\right\} \quad t=1,2,\ldots \tag{1.5.3}$$

1.6 Equilibrium combinations.

For the LP-problem, solutions $(\tilde{x},\tilde{y}) \in R_+^{n+m}$, $(\tilde{u},\tilde{v}) \in R_+^{m+n}$ of the system:

$$\left.\begin{aligned}(B-\tfrac{1}{\rho}A)\tilde{x}+\tilde{y} &= f\\ (B'-\pi A')\tilde{u}-\tilde{v} &= p\\ \tilde{v}'\tilde{x}+\tilde{u}'\tilde{y} &= 0\end{aligned}\right\}, \tag{1.6.1}$$

are of special interest. For, putting the initial vector $x(0) := \tilde{x}$, it appears that $(x(t),y(t)) := \rho^t(\tilde{x},\tilde{y})$, $t=1,2,\ldots$ and $(u(t),v(t)) := \pi^t(\tilde{u},\tilde{v})$, $t=1,2,\ldots$ are P- and D-feasible solutions, which satisfy (1.5.1) and (1.5.2). Hence, when P- or D-directedness is assumed, these feasible solutions are optimal.

With respect to non-negative solutions of (1.6.1) - further to be called equilibrium combinations - some interesting properties are derived:

a) If the LP-problem is P- or D-directed and there is an inital vector $x(0) \in R_+^n$ for which the LP-problem is superregular, then there exists an equilibrium combination for this LP-problem.

b) If $(\tilde{x},\tilde{y}),(\tilde{u},\tilde{v})$ is an equilibrium combination of an LP-problem which is P- or D-directed and superregular, and if $((\tilde{x},\tilde{y}),(\tilde{u},\tilde{v}))$ satisfies a certain condition concerning non-degeneratedness [1], then for every P- and D-optimal

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1) This condition is slightly stronger [3,§8.1 and §9.12] than the requirement that, simultaneously, $(\tilde{x},\tilde{y})$ is unique given $(\tilde{u},\tilde{v})$ and, the other way round, $(\tilde{u},\tilde{v})$ is unique given $(\tilde{x},\tilde{y})$.

solution $(\hat{x}(t),\hat{y}(t))$, $t=1,2,\ldots$ and $(\hat{u}(t),\hat{v}(t))$, $t=1,2,\ldots$ resp.:

$$(\frac{1}{\rho})^t(\hat{x}(t),\hat{y}(t)) \to (\tilde{x},\tilde{y}),\ t\to\infty$$

$$(\frac{1}{\pi})^t(\hat{u}(t),\hat{v}(t)) \to (\tilde{u},\tilde{v}),\ t\to\infty$$

Moreover, the latter property, offers the possibility to construct a linear programming problem over a finite horizon, from which all optimal solutions of the original infinite horizon problem can be found.

1.7 The model of Hansen and Koopmans.

Hansen and Koopmans [4] give a model with a von Neumann type technology. There are L capital goods, M resources other than capital goods, K consumption goods and I linear productive processes with a constant technology over time. For each period t, we define:

- $z(t) \in R^L_+$: the capital inputs,
- $z(t+1) \in R^L_+$: the capital outputs,
- $w \in R^m_+$: the resource availabilities, which are supposed to be constant over time,
- $\tilde{y}(t) \in R^K_+$: the consumption,
- $\tilde{x}(t) \in R^I_+$: the activity levels of production.

We have the following relation between these quantities:

(capital input) $\quad \tilde{\tilde{A}}\tilde{x}(t) \leqq z(t) \quad$ (1.7.1)

(capital output) $\quad -\tilde{\tilde{B}}\tilde{x}(t) \leqq -z(t+1) \quad$ (1.7.2)

(resources) $\quad \tilde{\tilde{C}}\tilde{x}(t) \leqq w \quad$ (1.7.3)

(consumption) $\quad -\tilde{\tilde{D}}\tilde{x}(t) \leqq -\tilde{y}(t) \quad$ (1.7.4)

With respect to the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, the following assumptions are introduced:

a) $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \geqq 0$

b) $\sum_i \tilde{a}_{li}, \sum_i \tilde{b}_{li}, \sum_i \tilde{c}_{mi}, \sum_i \tilde{d}_{ki} > 0$, for all l, m, k

c) $\sum_l \tilde{a}_{li}, \sum_m \tilde{c}_{mi}, \sum_l b_{li} + \sum_k d_{ki} > 0$, for all i

The objective function is specified by the suppositions:

d) For each period, the same objective function $u(y)$ is applicable.

e) The objective function $u(y)$ is defined for each consumption y. Moreover, the objective function is concave, continuously differentiable and increasing with regard to each component y_j of y.

f) The objective function of a sequence of consumptions $\tilde{y}(1), \tilde{y}(2), \ldots$ is defined by:

$$\sum_{t=1}^{\infty} \alpha^t u(\tilde{y}(t)), \tag{1.7.5}$$

where α is a discount factor $0 < \alpha < 1$.

In order to applicate the properties mentioned in 1.4 to 1.6, it is necessary to linearize the objective function $u(y)$. For the sake of simplicity we assume that, with the help of a suitable transformation of the consumption vector, the objective function (1.7.5) can be approximated sufficiently by:

$$\sum_{t=1}^{\infty} \alpha^t [q'y^{(1)}(t) + r'y^{(2)}(t) + s'y^{(3)}(t)], \tag{1.7.6}$$

where $q > r > s > 0$ and where the transformed consumption vectors $y^{(1)}(t), y^{(2)}(t), y^{(3)}(t)$ are bounded by:

$$\left.\begin{array}{l} 0 \leqq y^{(1)}(t) \leqq v^{(1)} \\ 0 \leqq y^{(2)}(t) \leqq v^{(2)} \\ 0 \leqq y^{(3)}(t) \end{array}\right\} \quad t=1,2,... \tag{1.7.7}$$

Since the total consumption for period t is expressed by the sum $y^{(1)}(t)+y^{(2)}(t)+y^{(3)}(t)$, the consumption inequalities take the form:

$$-\tilde{D}x(t) \leqq -\tilde{G}y^{(1)}(t)-\tilde{G}y^{(2)}(t)-\tilde{G}y^{(3)}(t), \quad t=1,2,..., \tag{1.7.8}$$

where $\tilde{G}$ is a non-negative matrix, satisfying:

$$\sum_k \tilde{g}_{km} > 0, \text{ for all m.} \tag{1.7.9}$$

The inequalities (1.7.1) and (1.7.2) may be combined into the inequalities $\tilde{\tilde{A}}\tilde{\tilde{x}}(t+1)-\tilde{\tilde{B}}\tilde{\tilde{x}}(t) \leqq 0$, $t=0,1,2,...$, so that the capital inputs/outputs $z(t)$ are eliminated. With this simplification, the inequalities (1.7.1), (1.7.2), (1.7.3), (1.7.7), and (1.7.8) may be written in the form:

$$\begin{bmatrix} \tilde{A} & 0 & 0 & 0 \\ \tilde{C} & 0 & 0 & 0 \\ -\tilde{D} & \tilde{G} & \tilde{G} & \tilde{G} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t+1) \\ y^{(1)}(t+1) \\ y^{(2)}(t+1) \\ y^{(3)}(t+1) \end{bmatrix} - \begin{bmatrix} \tilde{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ y^{(1)}(t) \\ y^{(2)}(t) \\ y^{(3)}(t) \end{bmatrix} \leqq \begin{bmatrix} 0 \\ w \\ 0 \\ v^{(1)} \\ v^{(2)} \end{bmatrix} \tag{1.7.10}$$

Defining the quantities of the LP-problem in §1.1 as follows:

$$B := \begin{bmatrix} \tilde{A} & 0 & 0 & 0 \\ \tilde{C} & 0 & 0 & 0 \\ -\tilde{D} & \tilde{G} & \tilde{G} & \tilde{G} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \quad A := \begin{bmatrix} \tilde{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x(t) := \begin{bmatrix} \tilde{x}(t) \\ y^{(1)}(t) \\ y^{(2)}(t) \\ y^{(3)}(t) \end{bmatrix} \quad f := \begin{bmatrix} 0 \\ w \\ 0 \\ v^{(1)} \\ v^{(2)} \end{bmatrix} \quad p := \begin{bmatrix} 0 \\ q \\ r \\ s \end{bmatrix}$$

and putting $\pi := \alpha$, $\rho := 1$, it appears that this linearized model of Hansen and Koopmans can be written in the LP-model of (1.1.1). The role of the initial capital stocks $z(1)$ is taken over by the initial vector $x(0)$.

We observe that both conditions of directedness (§1.2) are satisfied.

Now, we shall compare some terms and conditions of Hansen and Koopmans, to some given in §1.2 to §1.4.

g) The viability condition of Hansen and Koopmans requires the existence of an $x \geqq 0$ satisfying $\tilde{\tilde{B}}x > \tilde{\tilde{A}}x > 0$. This concept is strongly related to the P-regularity condition (1.4-a), viz.: if $x \geqq 0$ satisfies $(B-A)x < 0$, then $(\tilde{B},0,0,0)x > (\tilde{A},0,0,0)x \geqq 0$.

h) Hansen and Koopmans call an initial capital stock z in (1.7.1) more-than-reproducible if a sequence of non-negative vectors $(\tilde{x}(1),z(1)),(\tilde{x}(2),z(2)),\dots,(\tilde{x}(T),z(T))$ exists which satisfies (1.7.1) to (1.7.4) and for which $z(1) \leqq z \leqq z(T)$, $z(1) < z(T)$. The existence of such a z is implied by the existence of an $x(0) \geqq 0$ for which the LP-problem is P-regular. For, the latter implies (see 1.4-a with $\rho = 1$) the existence of an $x > 0$ and a scalar $\varepsilon > 0$ such that $[(1+\varepsilon)B-A]x < 0$. From this, the existence of a more-than-reproducible capital stock can be easily deduced.

i) To find an invariant optimal capital stock, which is in fact the primal part of our equilibrium combination, Hansen and Koopmans introduce the growth capability condition. This condition requires the existence of an $\bar{x} \geqq o$ satisfying $(\tilde{A}-\alpha\tilde{B})\bar{x} < 0$, α being the discount factor in (1.7.5). Clearly, in connection with the definitions (1.7.11) and the non-negativity assumptions (1.7-a,b,c), this growth capability condition is equivalent to the condition that system (1.3.1) is solvable; i.e. that the primal part of the superregularity condition is satisfied.

Now, we shall attend to the dual part of the LP-problem. Since for all j: $\sum_m c_{mj} > 0$ and for all n: $\sum_i g_{in} > 0$, definition (1.7.11) implies that the system $(B-A)x \leqq 0$ has no solution $x \geq 0$. By virtue of Stiemke's theorem [1], the latter implies the existence of a $u \geqq 0$ such that $(B'-A')u > 0$. Clearly, we may conclude that the system

$$(B'-A')u > p,\ u \geqq 0,$$

is solvable for every p.
Since in this case $\rho := 1$, it appears that the dual part of the superregularity condition (1.3.2) is satisfied. Moreover, with the help of 1.4-e, we also may conclude that the system

$$(B'-\pi A')u > p,\ u \geqq 0,$$

is solvable for every $\pi \in [0,1]$. This means that the D-regularity condition, too, is satisfied.
Thus, we find that all conditions of §1.2 to §1.6 are satisfied. This means that all properties of these paragraphs are valid for the linearized Hansen and Koopmans model.

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1) Stiemke's theorem inequality form: a system of linear equalities $Ax \leq 0$ has a solution $x \geq 0$, if and only if the system $A'u > 0$ has no solution $u \geqq 0$.

2 LEMKE'S COMPLEMENTARITY ALGORITHM FOR THE CALCULATION OF EQUILIBRIUM COMBINATIONS.

2.1 Introduction

In §1.6, equilibrium combinations are defined as solutions $(x,y) \in R_+^{n+m}$, $(u,v) \in R_+^{m+n}$ of the system

$$\left.\begin{aligned} (B-\tfrac{1}{\rho}A)x + y &= f \\ (B'-\pi A')u - v &= p \\ v'x + u'y &= 0 \end{aligned}\right\} \tag{2.1.1}$$

With the help of Kakutani's fixed point theorem, it can be shown [3,chapter 8] that if the LP-problem is superregular and P- or D-directed, an equilibrium combination exists. Now, it is our aim to demonstrate that, for such an LP-problem, it is possible with the help of Lemke's complementarity algorithm [5] to calculate an equilibrium combination.

First we will show that the P- or D-directedness condition offers the possibility to write (2.1.1) in the following complementarity form:

$$\begin{bmatrix} 0 & G \\ G'+C' & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} y \\ -v \end{bmatrix} = \begin{bmatrix} f \\ p \end{bmatrix} \tag{2.1.2}$$

C being a non-negative matrix.

Next, it will be shown that, although the usual matrix conditions for the application of the Lemke algorithm on problem (2.1.2) are not satisfied, this algorithm does, indeed, give the desired results [1].

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[1] This result is based on unpublished investigations (done in august and september 1973) of mr. de Beer, fellow of the computer center of the Tilburg School of economies. In october 1973, Dantzig and Manne [1] published a similar result. However, they require non-negativity of at least one of the matrixes A and B which is stronger than the requirement of P- or D-directedness.

2.2 Symmetry in the primal and dual part of system (2.1.1)

First we wish to note that (2.1.1) may also be written:

$$\left.\begin{aligned} (A'-\tfrac{1}{\pi}B')u + v &= (-p) \\ (A-\rho B)x - y &= (-f) \\ x'v + y'u &= 0 \end{aligned}\right\} . \qquad (2.2.1)$$

A similar relation of symmetry is also present in the definitions of P- and D-directedness. This means that the treatment of system (2.1.1) on the assumption of P-directedness is also applicable on system (2.2.1) on the assumption of D-directedness. So, all results derived when P-directedness is assumed, are also valid when the LP-problem is assumed to be D-directed.

2.3 Transformation into the complementarity form

In case the LP-problem is P-directed, we define three matrices: C(mxn), D(diagonal mxm), and G(mxn):

$$\left.\begin{aligned} c_{i.} &:= (\tfrac{1}{\rho\pi}-1)b_{i.} && \text{if } b_{i.} \geqq 0 \\ c_{i.} &:= (\tfrac{1}{\rho}-\pi)a_{i.} && \text{if } b_{i.} \ngeqq 0 \end{aligned}\right\} \qquad (2.3.1)$$

$$\left.\begin{aligned} d_{ii} &:= \tfrac{1}{\rho\pi} && \text{if } b_{i.} \geqq 0 \\ d_{ii} &:= 1 && \text{if } b_{i.} \ngeqq 0 \end{aligned}\right\} \qquad (2.3.2)$$

$$G := (B-\tfrac{1}{\rho}A). \qquad (2.3.3)$$

These definitions imply

$$D(B-\pi A) = G + C, \qquad (2.3.4)$$

and, in connection with the definition of P-directedness (§1.2), also the non-negativity of matrix C. With these definitions, we find the following relations between system (2.1.1) and system (2.1.2).

2.4 Proposition

a) A non-negative solution $(\hat{x},\hat{y},\hat{u},\hat{v})$ of (2.1.2) satisfies $(\hat{u},\hat{x})'(\hat{y},\hat{v}) = 0$ if and only if $(\bar{x},\bar{y}) := (\hat{x},\hat{y}), (\bar{u},\bar{v}) := (D\hat{u},\hat{v})$ is an equilibrium combination of the LP-problem which is P-directed.

b) If the LP-problem is P-directed and superregular, then the systems

$$(G+C)x < f,\ x \geqq 0, \tag{2.4.2}$$

$$G'u > p,\ u \geqq 0, \tag{2.4.3}$$

are solvable.

Proof

(a): The validity of this property follows immediately from (2.3.2) to (2.3.4) and from the definition of an equilibrium combination §1.6.

(b): When the LP-problem is superregular then the systems

$$(B-\pi A)x < f,\ x \geqq 0, \tag{2.4.4}$$

$$(B'-\frac{1}{\rho}A')u > p,\ u \geqq 0, \tag{2.4.5}$$

are solvable. Since P-directedness (§1.2) requires the non-negativity of components f_i in case the corresponding row-vector $b_{i.}$ contains an negative component, the definition of matrix D and the inequalities $0 < \rho\pi < 1$

imply:

$$Df \leqq \left(\frac{1}{\rho\pi}\right) f. \tag{2.4.6}$$

From this inequality and from the solvability of (2.4.4) we may conclude that $D(B-\pi A)x < \left(\frac{1}{\rho\pi}\right) f$ possesses a non-negative solution. This means, in connection with (2.3.9), that (2.4.2) is indeed solvable.
The solvability of (2.4.3) is implied immedeately by definition (2.3.3) and the solvability of (2.4.5).

2.5 Geometric description of Lemke's complementarity algorithm

Proposition 2.4-a states that from non-negative solutions $(\hat{u},\hat{y},\hat{u},\hat{v})$ of system (2.1.2) which satisfy $(\hat{u},\hat{x})'(\hat{y},\hat{v}) = 0$ all equilibrium combinations can be found. It will be shown that solutions of this type - further to be called complementary solutions - can be calculated by Lemke's complementarity algorithm.
This algorithm is based on the following set in $R^{n+m+m+n+1}$:

$$XYUV\Gamma: = \left\{ (x,y,u,v,\gamma) \in R_+^{n+m+m+n+1} \;\middle|\; \begin{array}{r} Gx+y-\gamma r=f \\ (G'+C')u-v+\gamma s=p \end{array} \right\}, \tag{2.5.1}$$

where $r \in R^m$ and $s \in R^n$ are any positive vectors.
With respect to vectors of $XYUV\Gamma$ the following denominations are used:

a) $(x,y,u,v,\gamma) \in XYUV\Gamma$ is called almost complementary, if:

$$(v,u)'(x,y) = 0, \tag{2.5.2}$$

b) an almost complementary point (x,y,u,v,γ) is called complementary if $\gamma = 0$.

Clearly, since the vectors r and s are positive, there is a $\underline{\gamma} \geqq 0$ such that all points of the half line:

$$\left.\begin{aligned} (x,u) &:= 0 \\ (v,y) &:= \gamma(s,r) + (-p,f) \\ \gamma &\geqq \underline{\gamma} \end{aligned}\right\} \qquad (2.5.3)$$

are almost complementary.

Starting from this half line, the algorithm follows a path of almost complementary points, generated by extreme points of the set XYUVΓ. After having passed a finite number of almost complementary extreme points, the algorithm will stop in one of two ways: (1) an almost complementary point with $\gamma = 0$ is reached (i.e.: a complementary point is found); or (2) another halfline is found. It is our aim to prove that, if the systems (2.4.2) and (2.4.3) are solvable (i.e.: the LP-problem is superregular), (2.5.3) is the only halfline of almost complementary points, so that the second way is excluded.

To avoid cycling, it is assumed that all extreme points of XYUVΓ which satisfy (2.5.2), are non-degenerated; i.e.: such a point $(\hat{x},\hat{y},\hat{u},\hat{v},\hat{\gamma})$ contains exactly (m+n) positive components. On this assumption it can be shown that, the set of almost complementary points consists of polygonal paths, without cycles or branches, going though the almost complementary extreme points of XYUVΓ. If (2.5.3) is the only halfline then the polygonal path which contains these halfline, terminates in a complementary point of XYUVΓ.

2.6 Proposition

If the systems

$$(G+C)x < f, \; x \geqq 0, \qquad (2.6.1)$$

$$G'u > p,\ u \geqq 0, \tag{2.6.2}$$

C being a non-negative matrix, are solvable, then the half-line defined by (2.5.3) is the only halfline of almost complementary points of XYUVΓ.

Proof

Every halfline of almost complementary points can be represented by:

$$(\hat{x},\hat{y},\hat{u},\hat{v},\hat{\gamma}) + \lambda(\tilde{x},\tilde{y},\tilde{u},\tilde{v},\tilde{\gamma}), \tag{2.6.3}$$

where $(\hat{x},\hat{y},\hat{u},\hat{v},\hat{\gamma})$ is an almost complementary point of XYUVΓ, and $(\tilde{x},\tilde{y},\tilde{u},\tilde{v},\tilde{\gamma})$ a non-negative solution of the homogeneous system:

$$\left.\begin{aligned} Gx+y-\gamma r &= 0 \\ (G'+C')u-v+\gamma s &= 0 \end{aligned}\right\}. \tag{2.6.4}$$

Since, (2.6.3) is supposed to be almost complementary, definition 2.5-a implies:

$$(\hat{y},\hat{x})'(\tilde{u},\tilde{v}) = 0, \tag{2.6.5}$$

$$(\hat{v},\hat{u})'(\tilde{x},\tilde{y}) = 0, \tag{2.6.6}$$

$$(\tilde{v},\tilde{u})'(\tilde{x},\tilde{y}) = 0. \tag{2.6.7}$$

From (2.6.4) and (2.6.7) we may deduce

$$\left.\begin{aligned} \tilde{u}'G\tilde{x} - \tilde{\gamma}\tilde{u}'r &= 0 \\ \tilde{u}'(G+C)\tilde{x} + \tilde{\gamma}\tilde{x}'s &= 0 \end{aligned}\right\}.$$

Combining these equalities, we find:

$$\tilde{u}'C\tilde{x} + \tilde{\gamma}(r'\tilde{u}+s'\tilde{x}) = 0. \tag{2.6.8}$$

Since C, r, s, $\tilde{\gamma}$, $\tilde{x}$, $\tilde{u} \geqq 0$, equality (2.6.8) implies:

$$\tilde{\gamma}(r'\tilde{u}+s'\tilde{x}) = 0. \tag{2.6.9}$$

Now, suppose that (2.6.3) is a different halfline from (2.5.3). Then, $(\tilde{u},\tilde{v}) \geq 0$. Since r, s > 0, this implies by virtue of (2.6.9): $\tilde{\gamma} = 0$. Moreover, since $(\tilde{x},\tilde{y},\tilde{u},\tilde{v},\tilde{\gamma})$ is a solution of (2.6.4) we may conclude:

$$G\tilde{x}+\tilde{y} = 0, \tag{2.6.10}$$

$$(G'+C')\tilde{u}-\tilde{v} = p, \tag{2.6.11}$$

$$(\tilde{u},\tilde{x}) \geq 0. \tag{2.6.12}$$

Now, we distinguish two cases: (a) $\tilde{x} \geq 0$ and (b) $\tilde{u} \geq 0$.

(a) Since $(\hat{x},\hat{y},\hat{u},\hat{v},\hat{\gamma}) \in XYUV\Gamma$, definition (2.5.1) implies:

$$G'\hat{u}-\hat{v} = p-\hat{\gamma}s-C'\hat{u}. \tag{2.6.13}$$

Combining (2.6.10), (2.6.13), and (2.6.6) we find:

$$(p'-\hat{\gamma}s'-\hat{u}'C)\tilde{x} = \hat{u}'(G\tilde{x}+\tilde{y}) = 0.$$

Since $\hat{u}$, C, $\tilde{x}$, $\hat{\gamma}$, s $\geqq 0$ this implies $p'\tilde{x} \geqq 0$. Thus, we have found that $\tilde{x}$ satisfies:

$$\left.\begin{aligned} G\tilde{x} &\leqq 0 \\ p'\tilde{x} &\geqq 0 \\ \tilde{x} &\geq 0 \end{aligned}\right\}$$

By virtue of Stiemke's theorem, this implies that system (2.6.2) is non-solvable.

(b) Since $(\hat{x},\hat{y},\hat{u},\hat{v},\hat{\gamma}) \in XYUV\Gamma$, definition (2.5.1) implies:

$$(G+C)\hat{x}+\hat{y} = f+\hat{\gamma}r+C\hat{x} \qquad (2.6.14)$$

With the help of (2.6.11), (2.6.14) and (2.6.5), it can, in a similar way as for case (a), be shown that now the non-solvability of (2.6.1) is implied.

Hence we may conclude: the supposition that the halfline represented by (2.6.3) is not the same as (2.5.3), is contradictory with respect to the assumption that the systems (2.6.1) and (2.6.2) are both solvable.

2.6 Conclusion

Combining proposition 2.4 and 2.6 we may conclude: If the LP-problem is P-directed and superregular, then with the help of Lemke's complementarity algorithm, an equilibrium combination can be found. Since the LP-problem is symmetric in the primal and dual part, as mentioned in §2.2, this is also possible if the LP-problem is D-directed and superregular.

G.W.

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